PHYSICAL REVIEW E **70**, 017101 (2004)

Deformation of a renormalization-group equation applied to infinite-order phase transitions

Hisamitsu Mukaida*
Department of Physics, Saitama Medical College, 981 Kawakado, Iruma-gun, Saitama, 350-0496, Japan
(Received 24 August 2003; published 7 July 2004)

By adding a linear term to a renormalization-group equation in a system exhibiting infinite-order phase transitions, the asymptotic behavior of running coupling constants is derived in an algebraic manner. A benefit of this method is presented explicitly using several examples.

DOI: 10.1103/PhysRevE.70.017101 PACS number(s): 64.60.Ak, 05.70.Fh, 05.70.Jk, 11.10.Hi

I. INTRODUCTION

The renormalization-group (RG) technique is one of the most powerful methods for investigating critical phenomena in statistical physics [1]. In general, RG transformation (RGT) consists of a coarse graining and a rescaling. It reduces many-body effects in a statistical model to an ordinary differential equation of coupling constants. The differential equation is called the RG equation (RGE), and generally has the following form:

$$\frac{d\boldsymbol{g}}{dt}=\boldsymbol{V}(\boldsymbol{g}), \tag{1}$$

where $\boldsymbol{g}=(g_1,\ldots,g_n)$, a collection of coupling constants depending on t, and $t=\ln L$ with L giving the length scale of the coarse graining in the RG. One obtains a beta function $\boldsymbol{V}(\boldsymbol{g})=[V_1(\boldsymbol{g}),\ldots,V_n(\boldsymbol{g})]$ by applying the RGT explicitly to a statistical model. We can derive universal exponents that characterize critical phenomena from asymptotic behavior of solutions of Eq. (1) for large t.

Since the asymptotic behavior is determined by vicinity of a fixed point $\boldsymbol{g}^*$, linearization of $\boldsymbol{V}(\boldsymbol{g})$ about $\boldsymbol{g}^*$ is effective enough to obtain the exponents. For example, in a second-order phase transition, the correlation length ξ typically behaves as

$$\xi=\text{const}/|T-T_c|^{\nu}, \tag{2}$$

where ν is the correlation-length exponent and T is a parameter specifying a state in a statistical model (e.g., the temperature). In the language of RG, T parametrizes initial values of RGE. The trajectory starting from the initial value at $T=T_c$ is absorbed into the fixed point. Other trajectories approach $\boldsymbol{g}^*$ once but leave the fixed point subsequently, as shown in Fig. 1. This implies that it takes longer for $\boldsymbol{g}$ to leave the fixed point as T approaches T_c. If the scaling matrix $M(\boldsymbol{g}^*)$, where

$$M_{ij}(\boldsymbol{g}^*)\equiv\frac{\partial V_i}{\partial g_j}(\boldsymbol{g}^*) \tag{3}$$

has a unique positive eigenvalue α, then this period behaves as $\text{const}\times(T-T_c)^{-\alpha}$. The universal exponent ν is obtained from α by $\nu=1/\alpha$. Thus we do not need to find an explicit solution of generally nonlinear RGE (1).

On the other hand, in the case of infinite-order phase transitions, ξ behaves as

$$\xi=\text{const}\times\exp(A/|T-T_c|^{\sigma}), \tag{4}$$

where σ is a universal exponent and A is a nonuniversal constant. Such behavior is observed when all the coupling constants are marginal, i.e., the canonical dimensions of the coupling constants are zero at $\boldsymbol{g}^*$. Since the linear term in $\boldsymbol{V}(\boldsymbol{g})$ is proportional to the canonical dimensions of $\boldsymbol{g}$, $M_{ij}(\boldsymbol{g}^*)=0$ for all i and j. It indicates that we cannot extract the asymptotic behavior from the scaling matrix $M(\boldsymbol{g}^*)$ in an infinite-order phase transition, in contrast to a second-order one. Therefore, explicit solutions were traditionally required in the case of an infinite-order phase transition such as the BKT phase transition [2].

This difficulty has been recently overcome in Ref. [3], where an RG for RGE (1) is used for deriving asymptotic behavior of solutions. A general idea of RG, applied as a tool for asymptotic analysis of nonlinear differential equations, is developed in Refs. [4,5].

In this report, we present another method. Namely, we derive σ from the following deformed RGE:

$$\frac{d\boldsymbol{g}}{dt}=\boldsymbol{\epsilon}(\boldsymbol{g}-\boldsymbol{g}^*)+\boldsymbol{V}(\boldsymbol{g})\equiv\overline{\boldsymbol{V}}(\boldsymbol{g}), \tag{5}$$

where $\boldsymbol{\epsilon}$ is a real number but not necessarily small. As we will see in the next section, the RG equation (7) for the RGE (1) has a complicated form compared with the deformed RGE. Hence, using the deformed RGE makes derivation of the critical exponent simple. Another benefit of this approach is as follows: suppose that an infinite-order phase transition

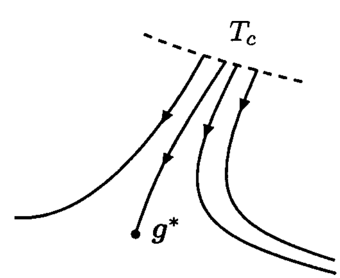

FIG. 1. Typical RG trajectories near a phase transition. As T changes, an initial value moves on the dashed line. The trajectory with $T=T_c$ is absorbed into the fixed point.

*Electronic address: mukaida@saitama-med.ac.jp

occurs when the spatial dimensions of the original statistical model are d_c. Then, the deformed RGE can be derived when they are $d_c-\epsilon$, under the condition that all the coupling constants have a common canonical dimension. This condition is satisfied by various field-theoretical models, e.g., an effective theory of antiferromagnets [6], a model containing several gauge fields [7], a model describing true self-avoiding random walks [8], and a model of nematic elastomers [9]. In Ref. [9], infrared asymptotic behavior in d_c dimensions and that in $d_c-\epsilon$ dimensions are analyzed separately because of the problem of the vanishing scaling matrix explained above. Our method enables us to obtain universal quantities in both of the cases simultaneously. We will show this advantage in the last example of Sec. IV.

II. RGE FOR RGE

Here we summarize the results of Ref. [3] that will be used later. We consider an RGE (1) for infinite-order phase transitions that are controlled by a fixed point $\boldsymbol{g}^*$. In what follows, we put $\boldsymbol{g}^*=\boldsymbol{0}$ for convenience. Suppose that we have obtained $\boldsymbol{V}(\boldsymbol{g})$ by the lowest-order perturbation. Since linear terms vanish in infinite-order phase transitions, components of $\boldsymbol{V}(\boldsymbol{g})$ are quadratic in $\boldsymbol{g}$. Hence the scaling property

$$\boldsymbol{V}(k\boldsymbol{g})=k^2\boldsymbol{V}(\boldsymbol{g}) \tag{6}$$

holds in this case. The algebraic method to compute σ shown in Ref. [3] employs another RG than Eq. (1), which is defined as follows: let $\boldsymbol{g}(t,\boldsymbol{a}_0)$ be the solution of Eq. (1) with the initial condition $\boldsymbol{g}(0,\boldsymbol{a}_0)=\boldsymbol{a}_0$. Choose a real number τ and we evolve $\boldsymbol{g}$ in time by $s(\tau)$, such that $e^\tau\boldsymbol{g}(s(\tau),\boldsymbol{a}_0)\in S$, where S is a sphere with the radius $a_0\equiv|\boldsymbol{a}_0|$ and with the center at the origin. Thus we have the map $R_\tau:\boldsymbol{a}_0\mapsto e^\tau\boldsymbol{g}(s(\tau),\boldsymbol{a}_0)\equiv\boldsymbol{a}(\tau)$. Thanks to Eq. (6), R_τ satisfies the semigroup property $R_{\tau+\tau'}=R_\tau R_{\tau'}$, so that R_τ is called RG for RGE (1). The infinitesimal transformation leads to the following new RGE:

$$\frac{d\boldsymbol{a}}{d\tau}=\beta(\boldsymbol{a})\equiv-\frac{P(\boldsymbol{a})\boldsymbol{V}(\boldsymbol{a})}{\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})}a_0^2, \tag{7}$$

where $P(\boldsymbol{a})$ is the projection operator defined by $P_{ij}(\boldsymbol{a})=\delta_{ij}-a_ia_j/a_0^2$. Since $\beta(\boldsymbol{a})$ is perpendicular to $\boldsymbol{a}$, solutions of the new RGE are restricted on S. Introducing the polar coordinates $\{\theta_\alpha\}_{1\leq\alpha\leq n-1}$ on S and the corresponding orthonormal basis

$$\tilde{\boldsymbol{e}}_\alpha\equiv f_\alpha(\boldsymbol{a})^{-1}\frac{\partial\boldsymbol{a}}{\partial\theta_\alpha}, f_\alpha(\boldsymbol{a})\equiv\left|\frac{\partial\boldsymbol{a}}{\partial\theta_\alpha}\right|, \tag{8}$$

we can expand $\beta(\boldsymbol{a})$ as

$$\beta(\boldsymbol{a})=\sum_{\alpha=1}^{n-1}\tilde{\beta}_\alpha(\boldsymbol{a})\tilde{\boldsymbol{e}}_\alpha. \tag{9}$$

The new RGE is represented as

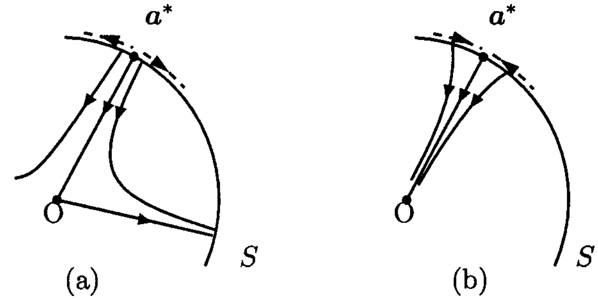

FIG. 2. Schematic trajectories of RGEs. The solid lines are for the original RGE (1) while the dashed lines are for the new RGE (7) defined on S. Here (a) is the case where a unique positive eigenvalue exists in $\mu(\boldsymbol{a}^*)$. (b) The case where all the eigenvalues of $\mu(\boldsymbol{a}^*)$ are negative.

$$\frac{d\theta_\alpha}{d\tau}(\boldsymbol{a})=f_\alpha^{-1}(\boldsymbol{a})\tilde{\beta}_\alpha(\boldsymbol{a}) \tag{10}$$

by the polar coordinates.

It is easily found that $\boldsymbol{a}^*\in S$ is a fixed point of the new RGE (7) if $\boldsymbol{g}(t,\boldsymbol{a}^*)$ is a straight flow line. In particular, a fixed point on an incoming straight flow line satisfying $\boldsymbol{a}^*\cdot\boldsymbol{V}(\boldsymbol{a}^*)<0$ plays an important role because trajectories near this fixed point correspond to trajectories of Eq. (1) approaching $\boldsymbol{g}^*$. In contrast to the original RGE, we can generally linearize the new RGE about $\boldsymbol{a}^*$. In Ref. [3], it is shown that the scaling matrix of the new RGE

$$\mu_{\alpha\beta}(\boldsymbol{a}^*)\equiv f_\alpha^{-1}(\boldsymbol{a}^*)\frac{\partial\tilde{\beta}_\alpha}{\partial\theta_\beta}(\boldsymbol{a}^*) \tag{11}$$

plays a similar role to $M(\boldsymbol{g}^*)$ in the original RGE describing a second-order phase transition. Namely, if the matrix $\mu(\boldsymbol{a}^*)$ has a unique positive eigenvalue λ, in which typical trajectories of the original RGE are in Fig. 2(a), we can observe divergence of the correlation length by one-parameter tuning and

$$\sigma=\frac{1}{\lambda} \tag{12}$$

in Eq. (4). On the other hand, if all the eigenvalues of $\mu(\boldsymbol{a}^*)$ are negative, where typical trajectories are in Fig. 2(b), $\boldsymbol{g}(t,\boldsymbol{a}_0)$ behaves as

$$\boldsymbol{g}(t,\boldsymbol{a}_0)\sim\frac{1}{C(\boldsymbol{a}^*)t}\boldsymbol{e}^*. \tag{13}$$

In this formula, $\boldsymbol{e}^*\equiv\boldsymbol{a}^*/a_0$ and $C(\boldsymbol{g})$ is defined by the relation

$$C(\boldsymbol{g})|\boldsymbol{g}|^3=-\boldsymbol{g}\cdot\boldsymbol{V}(\boldsymbol{g}). \tag{14}$$

The asymptotic behavior in Eq. (13) is important for investigating finite size scaling in a statistical system, for example.

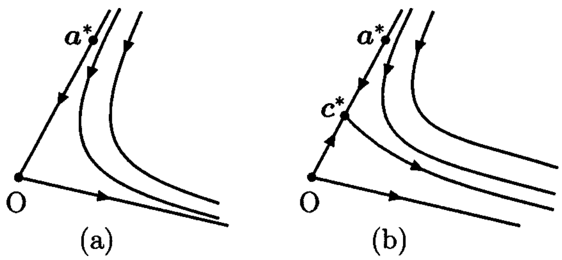

FIG. 3. (a) Schematic trajectories for the original RGE. (b) Those for the deformed RGE.

III. DEFORMED RGE

Next, we consider the deformed RGE (5) putting $\boldsymbol{g}^*=\boldsymbol{0}$. We can take $\epsilon>0$ without loss of generality. A fixed point $\boldsymbol{c}^*$ of the deformed RGE solves as

$$\bar{\boldsymbol{V}}(\boldsymbol{c}^*)=\epsilon\boldsymbol{c}^*+\boldsymbol{V}(\boldsymbol{c}^*)=\boldsymbol{0}. \tag{15}$$

A key feature of the deformed RGE is that $\boldsymbol{c}^*$ in Eq. (15) and a fixed point $\boldsymbol{a}^*$ of the new RGE (7) on an incoming straight flow line has one-to-one correspondence via

$$\boldsymbol{a}^*=\frac{a_0}{c^*}\boldsymbol{c}^* \tag{16}$$

as depicted in Fig. 3. Writing $\boldsymbol{V}(\boldsymbol{g})$ as

$$\boldsymbol{V}(\boldsymbol{g})=\sum_{\alpha=1}^{n-1}\tilde{V}_\alpha(\boldsymbol{g})\tilde{\boldsymbol{e}}_\alpha+\tilde{V}_n(\boldsymbol{g})\tilde{\boldsymbol{e}}_n, \tag{17}$$

where $\tilde{\boldsymbol{e}}_n\equiv\boldsymbol{g}/g$, we have the deformed RGE in the polar coordinates

$$\frac{d\theta_\alpha}{dt}(\boldsymbol{g})=f_\alpha^{-1}(\boldsymbol{g})\tilde{V}_\alpha(\boldsymbol{g}),$$

$$\frac{dg}{dt}(\boldsymbol{g})=\epsilon g+\tilde{V}_n(\boldsymbol{g}). \tag{18}$$

Expanding the above formula about the fixed point $\boldsymbol{c}^*$, we have the following scaling matrix $\bar{M}(\boldsymbol{c}^*)$:

$$\bar{M}_{\alpha\beta}(\boldsymbol{c}^*)=f_\alpha^{-1}(\boldsymbol{c}^*)\frac{\partial\tilde{V}_\alpha}{\partial\theta_\beta}(\boldsymbol{c}^*),$$

$$\bar{M}_{\alpha n}(\boldsymbol{c}^*)=f_\alpha^{-1}(\boldsymbol{c}^*)\frac{\partial\tilde{V}_\alpha}{\partial g}(\boldsymbol{c}^*),$$

$$\bar{M}_{n\alpha}(\boldsymbol{c}^*)=\frac{\partial\tilde{V}_n}{\partial\theta_\beta}(\boldsymbol{c}^*),$$

$$\bar{M}_{nn}(\boldsymbol{c}^*)=\left(\epsilon+\frac{\partial\tilde{V}_n}{\partial g}(\boldsymbol{c}^*)\right), \tag{19}$$

where α and β run from 1 to $n-1$. Since $\tilde{V}_\alpha(\boldsymbol{g})$ is a component perpendicular to $\boldsymbol{g}$, one finds that $\tilde{V}_\alpha(k\boldsymbol{c}^*)=0$ for all k with help of Eqs. (6) and (15). This means that $\partial_g\tilde{V}_\alpha(\boldsymbol{c}^*)=0$. On the other hand, $\partial_g\tilde{V}_n(\boldsymbol{c}^*)=2\tilde{V}_n(\boldsymbol{c}^*)/g^*=-2\epsilon$ because $\tilde{V}_n(\boldsymbol{g})$ is quadratic in g. Therefore,

$$\bar{M}_{\alpha n}=0,\quad \bar{M}_{nn}=-\epsilon \tag{20}$$

in Eq. (19). Furthermore, we can rewrite $\bar{M}_{\alpha\beta}(\boldsymbol{c}^*)$ in terms of $\mu_{\alpha\beta}(\boldsymbol{a}^*)$. In fact, $\mu(\boldsymbol{a}^*)$ in Eq. (11) is written as

$$\mu_{\alpha\beta}(\boldsymbol{a}^*)=f_\alpha^{-1}(\boldsymbol{a}^*)\frac{1}{C(\boldsymbol{a}^*)a_0}\frac{\partial\tilde{V}_\alpha}{\partial\theta_\beta}(\boldsymbol{a}^*). \tag{21}$$

Employing the scaling properties

$$C(k\boldsymbol{g})=C(\boldsymbol{g})$$

$$f_\alpha(k\boldsymbol{g})=kf_\alpha(\boldsymbol{g})$$

$$\frac{\partial\tilde{V}_\alpha}{\partial\theta_\beta}(k\boldsymbol{g})=k^2\frac{\partial\tilde{V}_\alpha}{\partial\theta_\beta}(\boldsymbol{g}), \tag{22}$$

we get

$$\bar{M}_{\alpha\beta}(\boldsymbol{c}^*)=\epsilon\mu_{\alpha\beta}(\boldsymbol{a}^*). \tag{23}$$

Equations (20) and (23) show that $\bar{M}(\boldsymbol{c}^*)$ has the form

$$\bar{M}(\boldsymbol{c}^*)=\left(\begin{array}{c|c} & 0 \\ \epsilon\mu(\boldsymbol{a}^*) & \vdots \\ & 0 \\ \hline *\cdots* & -\epsilon \end{array}\right) \tag{24}$$

in polar coordinates. It readily follows from this formula that $\bar{M}(\boldsymbol{c}^*)\tilde{\boldsymbol{e}}_n=-\epsilon\tilde{\boldsymbol{e}}_n$. Thus we can derive all the eigenvalues of $\mu(\boldsymbol{a}^*)$ from $\bar{M}(\boldsymbol{c}^*)$ by removing $-\epsilon$, which is the eigenvalue corresponding to the eigenvector $\tilde{\boldsymbol{e}}_n$, from the set of the eigenvalues of $\bar{M}(\boldsymbol{c}^*)$ and, by multiplying by $1/\epsilon$, the remaining eigenvalues. Further, if all the eigenvalues of $\bar{M}(\boldsymbol{c}^*)$ are negative, $\boldsymbol{g}(t,\boldsymbol{a}_0)$ behaves as

$$\boldsymbol{g}(t,\boldsymbol{a}_0)\sim\frac{1}{C(\boldsymbol{a}^*)t}\boldsymbol{e}^*=\frac{1}{\epsilon t}\boldsymbol{c}^*, \tag{25}$$

according to Eq. (13) and the scaling property of $C(\boldsymbol{a}^*)$ in Eq. (22).

IV. EXAMPLE

Here are several examples. The first example is taken from the two-dimensional classical *XY* model [2]. Here, the beta function $\boldsymbol{V}(\boldsymbol{g})$ is given as

$$V(g) = \begin{pmatrix} -g_2^2 \\ -g_1 g_2 \end{pmatrix}, \tag{26}$$

for $g_1, g_2 > 0$. The deformed RGE has the fixed point $c^* = (\epsilon, \epsilon)$. The scaling matrix $\bar{M}(c^*)$ of the deformed RGE is easily computed in terms of the Cartesian coordinates as

$$\bar{M}(c^*) = \begin{pmatrix} \epsilon & -2\epsilon \\ -\epsilon & 0 \end{pmatrix}. \tag{27}$$

It has the eigenvalues $-\epsilon$ and 2ϵ. Employing Eq. (12), we get

$$\sigma = \frac{\epsilon}{2\epsilon} = \frac{1}{2}, \tag{28}$$

which is a well-known result. As we have explained in the previous section, the other eigenvalue $-\epsilon$ always appears in a deformed RGE (5), which corresponds to the eigenvector c^*/c^*.

The next example is the RGE in a one-dimensional quantum spin chain, studied by Itoi and Kato [10]; it is defined by

$$V(g) = \begin{pmatrix} g_1(Ng_1 + 2g_2) \\ -g_2(2g_1 + Ng_2) \end{pmatrix}. \tag{29}$$

The deformed RGE has the following three nontrivial fixed points:

$$c_1^* = \left(-\frac{\epsilon}{N}, 0\right), \quad c_2^* = \left(0, \frac{\epsilon}{N}\right), \quad c_3^* = \frac{\epsilon}{N-2}(-1, 1). \tag{30}$$

The corresponding scaling matrices are

$$\bar{M}_1 = \begin{pmatrix} -\epsilon & -\dfrac{2\epsilon}{N} \\ 0 & \dfrac{N+2}{N}\epsilon, \end{pmatrix}, \quad \bar{M}_2 = \begin{pmatrix} \dfrac{N+2}{N}\epsilon & 0 \\ -\dfrac{2\epsilon}{N} & -\epsilon \end{pmatrix},$$

$$\bar{M}_3 = \begin{pmatrix} \dfrac{N\epsilon}{2-N} & \dfrac{2\epsilon}{2-N} \\ \dfrac{2\epsilon}{2-N} & \dfrac{N\epsilon}{2-N} \end{pmatrix}. \tag{31}$$

The eigenvalues of those matrices are, respectively,

$$\frac{N+2}{N}\epsilon, \quad \frac{N+2}{N}\epsilon, \quad \frac{2+N}{2-N}\epsilon, \tag{32}$$

up to the common eigenvalue $-\epsilon$. The other eigenvalues divided by ϵ are equal to those of the scaling matrices derived from the new RGE (7), which is computed in Ref. [3]. It should be noted that the deformed RGE's in the above two examples do not correspond to those in $2-\epsilon$ and $1-\epsilon$ dimensions, respectively. However, the derivation presented here is much simpler than the method using Eq. (7).

The last example is the RGE in a field-theoretical model for nematic elastomers, proposed in Ref. [9]. In contrast to the previous examples, the deformed RGE is obtained exactly in $3-\epsilon$ dimensions with

$$V(g) = \frac{-1}{8(4g_1 + g_2)} \begin{pmatrix} g_1(40g_1^2 + 68g_1 g_2 + 13g_2^2) \\ 2g_2(4g_1^2 + 32g_1 g_2 + 7g_2^2) \end{pmatrix}. \tag{33}$$

Although $V(g)$ is not quadratic polynomial, our result is applicable because all we need to apply the present method is the scaling property of $V(g)$, Eq. (6). The deformed RGE has the three fixed points

$$c_1^* = \left(\frac{4\epsilon}{5}, 0\right), \quad c_2^* = \left(\frac{4\epsilon}{59}, \frac{32\epsilon}{59}\right), \quad c_3^* = \left(0, \frac{4\epsilon}{7}\right). \tag{34}$$

One can check that the scaling matrices have the following respective eigenvalues:

$$4\epsilon/5, \quad -4\epsilon/59, \quad \text{and} \quad \epsilon/14 \tag{35}$$

in addition to the common eigenvalue $-\epsilon$. Now we turn to the case of just three dimensions. If $g_1, g_2 > 0$, infrared behavior of a system is governed by the fixed point c_2^* [9]. Since the eigenvalue at c_2^* is negative, $g(t, a_0)$ behaves as

$$g(t, a_0) \sim \frac{1}{\epsilon t} c_2^* = \frac{1}{t}\left(\frac{4}{59}, \frac{32}{59}\right) \tag{36}$$

for sufficiently large t, according to Eq. (25). The result is consistent with that in Ref. [9].

V. SUMMARY

We have shown how to derive asymptotic behavior of a solution of RGE for infinite-order phase transition, by adding a linear term to this RGE. This method can allow us to apply a result of the ϵ expansion to the case where $\epsilon = 0$.

[1] K.G. Wilson and J. Kogut, Phys. Rep., Phys. Lett. **12C**, 75 (1974).
[2] V.L. Berezinskiǐ, Zh. Éksp. Teor. Fiz. **59**, 907 (1970) [Sov. Phys. JETP **32**, 493 (1971)]; J.M. Kosterlitz and D.J. Thouless, J. Phys. C **6**, 1181 (1973).
[3] C. Itoi and H. Mukaida, Phys. Rev. E **60**, 3688 (1999).
[4] L.-Y. Chen, N. Goldenfeld, and Y. Oono, Phys. Rev. E **54**, 376 (1996).
[5] J. Bricmont and A. Kupiainen, Commun. Math. Phys. **150**, 193 (1992); J. Bricmont, A. Kupiainen, and G. Lin, Commun. Pure Appl. Math. **47**, 893 (1994).
[6] H. Hamidian, S. Jaimungal, G.W. Semenoff, P. Suranyi, and L.C.R. Wijewardhana, Phys. Rev. D **53**, 5886 (1996).
[7] J. Wirstam, J.T. Lenaghan, and K. Splittorff, Phys. Rev. D **67**, 034021 (2003).
[8] L. Peliti, Phys. Rep. **103**, 225 (1984).
[9] X. Xing and L. Radzihovsky, Europhys. Lett. **61**, 769 (2003).
[10] C. Itoi and M.H. Kato, Phys. Rev. B **55**, 8295 (1997).